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Kirchhoff scattering series for sources and receivers located in an elastic medium: Another insight into the concept of virtual events

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ABSTRACT

The convolution-type and correlation-type representation theorems are building blocks of wave-scattering theory whose usefulness expands in many seismological applications. For example, the Kirchhoff scattering series currently used for attenuating free-surface multiples has been derived from the convolution-type representation theorem. The recently introduced concept of virtual events, which allows us to put virtual sources and virtual receivers inside the subsurface based on the data collected at the sea surface, has been derived by a combined use of the convolution-type and correlation-type representation theorems. The formulation of inverse Kirchhoff scattering series and virtual events has been limited so far to the cases in which sources or receivers, or both, are located in the water. Unfortunately, this assumption is not valid, especially in the context of virtual events, in which both sources and receivers will often be located in a solid. We here redescribe the Kirchhoff scattering series and reformulate the concept of virtual events for the cases in which sources and receivers are in a solid. Moreover, we describe a new form of Kirchhoff series based on the correlation-type representation theorem and new formulae for computing virtual events which do not include the complex renormalization operation of the previous formulation.

1. Introduction

The key processes of seismic imaging include (i) removing from the seismic data all seismic events which contain at least one reflection at the free surface, (ii) removing all seismic events containing at least two reflections in the subsurface, and (iii) locating scattering point and reflection points in the subsurface. Ikelle and Gangi (2005), Ikelle (2006), and Ikelle et al. (in press) have proposed a framework for addressing these problems by using the Kirchhoff scattering series and the concept of virtual events. Their work is limited to the acoustic case. Our objective here is to extend the Kirchhoff scattering series and the formulation of the concept of virtual events to the elastic case.

The convolutive-type Kirchhoff scattering series (Ikelle et al., 2003) is probably the simplest way of deriving the process of free-surface multiples. By extending its formulation to the elastic case, we hope to facilitate its use for land seismics and OBS seismics. We also propose a new formulation of free-surface multiple attenuation using the correlation-type representation theorem. This reformulation turns to be very useful in the computation of virtual events. In particular, the complex renormalization technique proposed by Ikelle et al. (in press) can be replaced by a simple convolution of some of the terms of this series by one of the components of the actual data.

The concept of virtual events allows us to simulate seismic surveys with sources and receiver positions in the subsurface by using data recorded at the surface. In other words, we can create new seismic data from the recorded data which simulate, for example, scenarios in which sources are located below salt bodies. So the potentials of virtual events for improving seismic resolution are enormous. Therefore, it is important to extend the formulation of the concept of virtual events to the elastic case, as we will be dealing with virtual sources and receivers in solids in most practical cases.

In summary, the rest of our paper is divided into two sections. In the next section, we derive the convolutive-type Kirchhoff series for the stress and particle-velocity fields in the context of elastic media, with sources and receivers in solids. We will show how the acoustic solution derived by Ikelle et al. (in press) can be deduced as a particular case of our results. In the third section, we perform similar derivations to obtain the correlation-type Kirchhoff series of stress and particle-velocity fields. In this section, we also formulate the concept of virtual events for elastic media.

2. The convolutive-type Kirchhoff scattering series

Our objective in this section is to use the representation theorem to derive a Kirchhoff series for free-surface-multiple removal for the cases in which sources and receivers are located in a solid. Because we are focusing on the elastic case, it is useful to separate the Kirchhoff series of the stress field from that of the particle velocity. This is the approach that we will follow here. In each case, we will first establish an integral relationship between the pressure-field data containing all free-surface-related multiples and data without free-surface

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multiples, and second, we will expand this integral relationship in the form of a series that we will call the Kirchhoff series.

2.1. Demultiple series for the stress field

Although the stress field is not recorded in most seismic acquisitions today, recent trends in ocean-bottom seismics (OBS) and borehole seismics indicate that such measurements will be eventually available. Therefore, the formulae derived here represent potential applications.

Let us start by specifying the configuration of our problem. We consider a 3D model of the earth consisting of an inhomogeneous elastic solid, as described in Fig. 1. The position in this configuration is specified by the coordinate $\boldsymbol{x}=(x,y,z)=(x_1,x_2,x_3)$ with respect to a fixed Cartesian reference frame with the origin at O and three mutually perpendicular base vectors $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$. The unit vector $\mathbf{i}_3$ points vertically downward.

Let us now turn to the representation theorem (de Hoop, 1966, 1995; Gangi, 1970; Aki and Richards, 1980; Ikelle and Amundsen, 2005). We consider two geological models, one with the free surface and the other one without the free surface. We denote by $\tau_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and $\tilde{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ the stress fields, corresponding to the model with and without the free surface, respectively, for a source point at $\boldsymbol{x}_s$ and an observation at $\boldsymbol{x}$. According to Ikelle and Amundsen (2005), these two fields can be related as follows:

$$\begin{aligned}-\tau_{ij}(x',\omega,x_s) = -\tilde{\tau}_{ij}(x',\omega,x_s) + \oint_{x\in\partial D}\Delta_{klpq}n_k(x)\Big[-G_{ijpq}(x',\omega,x)\upsilon_l(x,\omega,x_s)\\ + i\frac{\Delta_{nlrs}}{\omega\rho(x)}\partial_n G_{ijrs}(x',\omega,x)\tau_{pq}(x,\omega,x_s)\Big]dS,\end{aligned} \tag{1}$$

where

$$\Delta_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{2}$$

and where $G_{ijpq}=G_{ijpq}(\boldsymbol{x}',\omega,\boldsymbol{x})$ is the green tensor of the model without a free surface and $\upsilon_l(\boldsymbol{x},\omega,\boldsymbol{x}_s)$ is the particle-velocity data corresponding to the medium with a free surface. The quantity $\rho(\boldsymbol{x})$ describes the density, and the vector $\boldsymbol{n}$ is the outward normal to ∂D. By assuming that $G_{ijpq}=G_{ijqp}$, Eq. (1) reduces to

$$\begin{aligned}-\tau_{ij}(x',\omega,x_s) = -\tilde{\tau}_{ij}(x',\omega,x_s) + \oint_{x\in\partial D}n_k(x)\Big[-G_{ijkl}(x',\omega,x)\upsilon_l(x,\omega,x_s)\\ + \frac{i}{\omega\rho(x)}\partial_n G_{ijnm}(x',\omega,x)\tau_{km}(x,\omega,x_s)\Big]dS.\end{aligned} \tag{3}$$

We can further simplify this equation by using the Sommerfeld radiation condition (Sommerfeld, 1954) to divide the surface bounding the domain D into two parts, ∂D_0 and ∂D_R, as depicted in Fig. 1. The surface ∂D_0 represents the free surface and ∂D_R represents the rest of the surface bounding the domain D. We can describe ∂D_R as a spherical surface with a radius R. By taking R to infinity, we can ignore the contribution along the surface ∂D_R of the Kirchhoff integral to (3) in accordance with the Sommerfeld's radiation condition. So (3) further reduces to

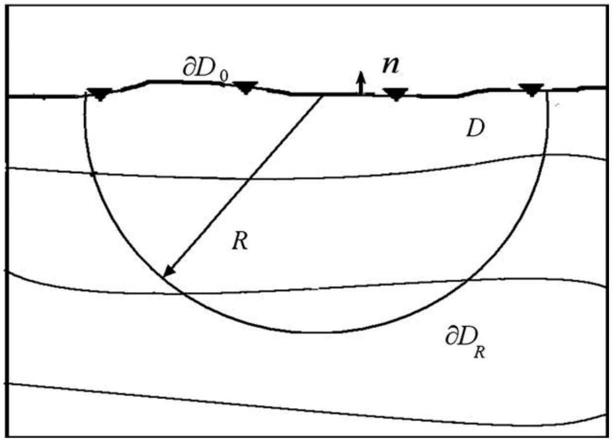

Fig. 1. Geometry of the physical and hypothetical seismic experiments. The surface $\partial D=\partial D_0+\partial D_R$ with an outward-pointing normal vector $\boldsymbol{n}$ encloses a volume D, the solid; ∂D_0 is a free surface with a vanishing stress field.

$$\tau_{ij}(x',\omega,x_s) = \tilde{\tau}_{ij}(x',\omega,x_s) + \int_{x\in\partial D_0}n_k(x)G_{ijkl}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS \tag{4}$$

or

$$\tau_{ij}(x',\omega,x_s) = \tilde{\tau}_{ij}(x',\omega,x_s) - h_{kl}(\omega)\int_{x\in\partial D_0}n_k(x)\,\tilde{\tau}_{ij}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS, \tag{5}$$

where $h_{kl}(\omega)$ denotes the inverted stress source.

By using the classical algebra of the geometrical series, we can reconstruct the stress field $\tilde{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ corresponding to the subsurface without a free surface from the observed stress field $\tau_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and the observed particle-velocity field $\upsilon_l(\boldsymbol{x},\omega,\boldsymbol{x}_s)$, as follows:

$$\tilde{\tau}_{ij}(x',\omega,x_s) = \sum_{n=0}\tau_{ij}^{(n)}(x',\omega,x_s), \tag{6}$$

where

$$\tau_{ij}^{(n+1)}(x',\omega,x_s) = -h_{kl}(\omega)\int_{x\in\partial D_0}n_k(x)\tau_{ij}^{(n)}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS, \tag{7}$$

$$\tau_{ij}^{(0)}(x',\omega,x_s) = \tau_{ij}(x',\omega,x_s). \tag{8}$$

It is interesting to observe that (6) reduces to the acoustic solution described by Ikelle et al. (2003) if we assume that $\tau_{ij}=-p\delta_{ij}$. Moreover, we can observe that each component of the stress field can be demultipled separately as long as all the components of the observed particle-velocity field are available.

2.2. Demultiple of the particle-velocity field

Let us denote by $\upsilon_i(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and $\tilde{\upsilon}_i(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ the particle-velocity fields corresponding to the model of the subsurface with and without a free surface, respectively. In accordance with the representation theorem, these two fields can be related as follows:

$$\begin{aligned}\upsilon_i(x',\omega,x_s) = \tilde{\upsilon}_i(x',\omega,x_s) + \oint_{x\in\partial D}n_m(x)\Big[-G_{ij}(x',\omega,x)\tau_{mj}(x,\omega,x_s)\\ + i\frac{c_{mpkl}(x)}{\omega}\partial_k G_{il}(x',\omega,x)\upsilon_p(x,\omega,x_s)\Big]dS\end{aligned} \tag{9}$$

and where $G_{ij}=G_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x})$ is the green tensor of the model without a free surface and where $\tau_{pq}(\boldsymbol{x},\omega,\boldsymbol{x}_s)$ is the stress data corresponding to the medium with a free surface. The quantity $c_{ijkl}(\boldsymbol{x})$ describes stiffnesses, and the vector $\boldsymbol{n}$ is the outward normal to ∂D. By again using the Sommerfeld radiation condition and the fact that stresses vanish at the free surface, (9) reduces to

$$\upsilon_i(x',\omega,x_s) = \tilde{\upsilon}_i(x',\omega,x_s) + i\oint_{x\in\partial D}\frac{c_{mpkl}(x)}{\omega}n_m(x)\partial_k G_{il}(x',\omega,x)\upsilon_p(x,\omega,x_s)dS \tag{10}$$

or

$$\upsilon_i(x',\omega,x_s) = \tilde{\upsilon}_i(x',\omega,x_s) - F_l(\omega)\oint_{x\in\partial D}n_l(x)\,\tilde{\tau}_{iq}(x',\omega,x)\upsilon_q(x,\omega,x_s)dS, \tag{11}$$

where $F_i(\omega)$ is a Fourier-transform of the inverted force-source signature. Notice that this equation relates the observed particle-velocity fields to the desired demultiple stress and velocity fields. This is significantly different from the results that we obtained earlier in (6), because we can demultiple the particle-velocity field only as a second step after all the components of the stress field have been demultipled. In other words, if $\tilde{\tau}_{il}(\boldsymbol{x}',\omega,\boldsymbol{x})$ is known, we can obtain the demultipled field as follows:

$$\tilde{\upsilon}_i(x',\omega,x_s)=\upsilon_i(x',\omega,x_s)+F_l(\omega)\int_{x\in\partial D_0}n_l(x)\,\tilde{\tau}_{iq}(x',\omega,x)\upsilon_q(x,\omega,x_s)dS. \tag{12}$$

One can alternatively implement the demultiple of the particle-velocity fields in terms of a series by substituting (6) in (10), We arrive at

$$\tilde{\upsilon}_i(x',\omega,x_s)=\sum_{n=0}\upsilon_i^{(n)}(x',\omega,x_s), \tag{13}$$

where

$$\upsilon_i^{(n+1)}(x',\omega,x_s)=F_l(\omega)\int_{x\in\partial D_0}n_l(x)\,\tilde{\tau}_{iq}^{(n)}(x',\omega,x)\upsilon_q(x,\omega,x_s)dS \tag{14}$$

and

$$\upsilon_i^{(0)}(x',\omega,x_s)=\upsilon_i(x',\omega,x_s). \tag{15}$$

Notice that we can now reconstruct the demultipled particle-velocity fields directly from the observed stress and velocity fields.

3. The correlation-type Kirchhoff scattering series and virtual events

Our objective in this section is also to derive the Kirchhoff series for predicting particle-velocity and stress fields corresponding to the model without a free surface. However, our starting point here will be the correlation representation theorem rather than the convolution representation theorem. For that reason, we end up with time-retarded particle-velocity and stress fields. Moreover, the resulting field will differ from those derived in the previous section because we assume that the contribution of retarded wavefields at infinity is negligible. However, we will show later in this section that we actually need these fields in order to properly predict our virtual events.

3.1. Demultiple series for the stress and particle-velocity fields

We consider two geological models, one with a free surface and the other without a free surface. We denote by $\tau_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and $\tilde{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ the stress fields corresponding to the model with and without a free surface, respectively, for a source point at $\boldsymbol{x}_s$ and an observation at $\boldsymbol{x}$. The time-retarded version of these fields will be denoted as $\bar{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and $\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)$. Actually, we will use the overline to denote the complex conjugate throughout the remainder of this paper. So the fields $\tau_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and $\tilde{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ can be related as follows (de Hoop, 1995; Bojarski, 1983):

$$\begin{aligned}-\tau_{ij}(x',\omega,x_s)\bar{a}_{ij}(\omega)={}&a_{ij}(\omega)\,\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)\\&+\bar{a}_{ij}(\omega)\oint_{x\in\partial D}\Delta_{klpq}n_k(x)\Big[-\overline{G}_{ijpq}(x',\omega,x)\upsilon_l(x,\omega,x_s)\\&+i\frac{\Delta_{lnrs}}{\omega\rho(x)}\partial_n\overline{G}_{ijrs}(x',\omega,x)\tau_{pq}(x,\omega,x_s)\Big]dS.\end{aligned} \tag{16}$$

We can simplify this equation by using the fact that the stress field is zero at the free surface:

$$\begin{aligned}-\tau_{ij}(x',\omega,x_s)\bar{a}_{ij}(\omega)={}&a_{ij}(\omega)\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)\\&-\bar{a}_{ij}(\omega)\int_{x\in\partial D_0}n_k(x)\overline{G}_{ijkl}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS\\&+\bar{a}_{ij}(\omega)\int_{x\in\partial D_R}n_k(x)\Big[-\overline{G}_{ijkl}(x',\omega,x)\upsilon_l(x,\omega,x_s)\\&+\frac{1}{\omega\rho(x)}\partial_n\overline{G}_{ijnq}(x',\omega,x)\tau_{kq}(x,\omega,x_s)\Big]dS.\end{aligned} \tag{17}$$

We can further simplify this equation by assuming that the stress field on the surface ∂D_R contains only downgoing waves and therefore does not contribute the data with receivers near the free surface. With this assumption, we arrive at

$$\begin{aligned}&a_{ij}(\omega)\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)-\bar{a}_{ij}(\omega)\int_{x\in\partial D_0}n_k(x)\overline{G}_{ijkl}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS\\&=-\tau_{ij}(x',\omega,x_s)\bar{a}_{ij}(\omega)\end{aligned} \tag{18}$$

or

$$\begin{aligned}&a_{ij}(\omega)\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)-\bar{h}_{kl}(\omega)\bar{a}_{ij}(\omega)\int_{x\in\partial D_0}n_k(x)\overline{\tilde{\tau}}_{ij}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS\\&=-\tau_{ij}(x',\omega,x_s)\bar{a}_{ij}(\omega),\end{aligned} \tag{19}$$

where $h_{ij}(\omega)$ is the inverse stress source, as defined in the previous section. Because we ignore the contribution of the Kirchhoff surface integral over $\partial D_{R\to\infty}$, the field $\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)$ that we can recover from (18) will not exactly be the time-retarded version of $\tilde{\tau}_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$. However, this retarded field will be used in our definition of virtual events later on.

By using the classical algebra of the geometrical series, as we did in the previous section, we can reconstruct the stress field $\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)$ corresponding to the subsurface without a free surface from the observed stress field $\tau_{ij}(\boldsymbol{x}',\omega,\boldsymbol{x}_s)$ and the observed particle-velocity field $\upsilon_l(\boldsymbol{x},\omega,\boldsymbol{x}_s)$ under the assumption that the Kirchhoff surface integral over $\partial D_{R\to\infty}$ is zero. We have

$$-a_{ij}(\omega)\overline{\tilde{\tau}}_{ij}(x',\omega,x_s)=\bar{a}_{ij}(\omega)\sum_{n=0}\tau_{ij}^{(n)}(x',\omega,x_s), \tag{20}$$

where

$$\tau_{ij}^{(n+1)}(x',\omega,x_s)=-h_{kl}(\omega)\int_{x\in\partial D_0}n_k(x)\tau_{ij}^{(n)}(x',\omega,x)\upsilon_l(x,\omega,x_s)dS, \tag{21}$$

$$\tau_{ij}^{(0)}(x',\omega,x_s)=\tau_{ij}(x',\omega,x_s). \tag{22}$$

It is interesting to observe that (6) reduces to the acoustic solution described by Ikelle and Amundsen (2005) if we assume that $\tau_{ij}=-p\delta_{ij}$. Moreover, we can observe that each component of the stress field can be demultipled separately as long as all the components of the observed particle-velocity field are available.

Let us now turn to the estimation of the time-retarded particle-velocity fields corresponding to the model without a free surface. Using exactly the same methodology as above, with the assumption that the Kirchhoff surface integral over $\partial D_{R\to\infty}$ is zero, we can show that

$$-\bar{b}_i(\omega)\,\overline{\tilde{\upsilon}}_i(x',\omega,x_s)=b_i(\omega)\sum_{n=0}\upsilon_i^{(n)}(x',\omega,x_s), \tag{23}$$

where

$$\upsilon_i^{(n+1)}(x',\omega,x_s)=\int_{x\in\partial D}\tilde{\tau}_{il}(x',\omega,x)\upsilon_l^{(n)}(x,\omega,x_s)dS \tag{24}$$

and

$$\upsilon_i^{(0)}(x',\omega,x_s)=\upsilon_i(x',\omega,x_s). \tag{25}$$

3.2. The convolution-correlation Kirchhoff series: virtual events and internal multiples

Fig. 2 shows how the convolutive-type Kirchhoff integral can be used to generate free-surface multiples as a multidimensional convolution of the events contained in our seismic data. Fig. 3 also shows that the correlation-type Kirchhoff integral can be used to generate free-surface multiples as a multidimensional correlation of the events contained in our seismic data. Moreover, we can also

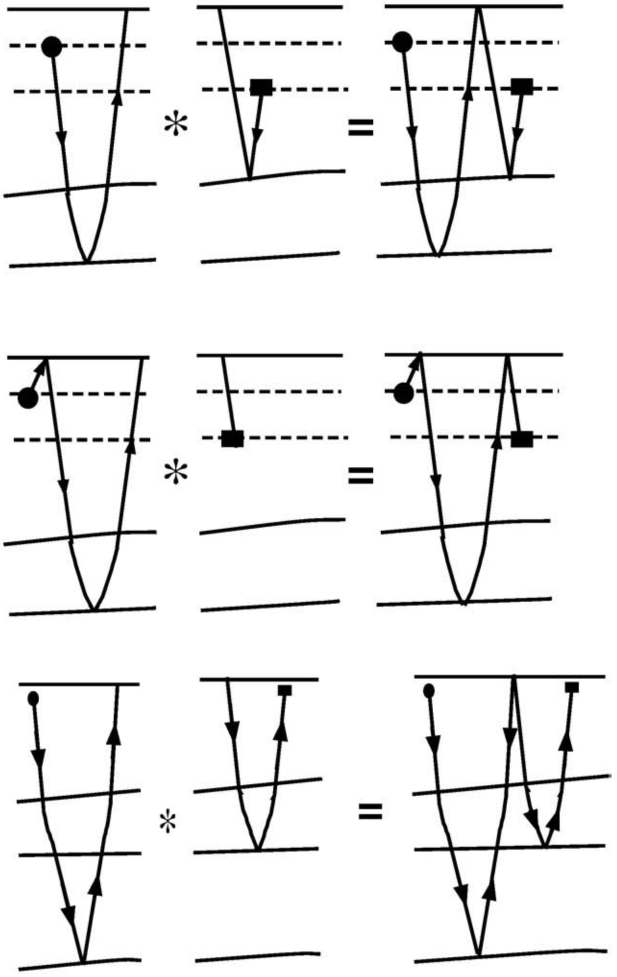

Fig. 2. An illustration of how the construction of free-surface multiples using the convolutive-type Kirchhoff integral. The symbol * denotes the multidimensional convolution operations.

generate internal multiples and primaries. However, it is generally difficult to generate only internal multiples with the correlation-type Kirchhoff integral. Ikelle and Gangi (2005) have proposed the construct in Fig. 4 for internal multiples. This construct introduces events which are not directly recorded in seismic data. They have named these events virtual events. Our objective is to describe how this construct can be extended to the elastic case. Moreover, because we will be using the time-retarded field derived from the Kirchhoff series in (20) instead of the complex conjugate of the observed field, our construct of internal multiples does not require the process of renormalization.

Following the line of derivations of Ikelle et al. (in press), we can express the stress tensor field of virtual events as follows:

$$-\tau^{V}_{ij}(x',\omega,x_{\mathrm{s}}) = -\int_{x\in\partial D_0} n_m(x)\overline{G}_{ijml}(x',\omega,x)\upsilon_l(x,\omega,x_{\mathrm{s}})dS \tag{26}$$

or

$$-\tau^{V}_{ij}(x_{\mathrm{r}},\omega,x_{\mathrm{s}}) = \overline{a}^{-1}_{ml}(\omega)\int_{x\in\partial D_0} n_m(x)\tilde{\tilde{\overline{\tau}}}_{ij}(x',\omega,x_{\mathrm{s}})\upsilon_l(x,\omega,x_{\mathrm{s}})dS. \tag{27}$$

In the acoustic media with a monopole source, (27) reduces to

$$P^{V}(x',\omega,x_{\mathrm{s}}) = \overline{s}(\omega)\int_{x\in\partial D_0} \tilde{P}(x',\omega,x)\upsilon_3(x,\omega,x_{\mathrm{s}})dS. \tag{28}$$

Let us now turn to the construction of internal multiples. From the description in Fig. 4, we can formulate the construction of internal multiples as follows:

$$-\tau^{IM}_{ij}(x_{\mathrm{r}},\omega,x_{\mathrm{s}}) = h_{ml}(\omega)\int_{x\in\partial D_0} n_m(x)\tau^{V}_{ij}(x_{\mathrm{r}},\omega,x)\upsilon_l(x,\omega,x_{\mathrm{s}})dS, \tag{29}$$

where $\tau^{IM}_{ij}(\boldsymbol{x}_r,\omega,\boldsymbol{x}_s)$ is a field of predicted internal multiples. Again, let us reiterate that the computation of $\tau^{IM}_{ij}(\boldsymbol{x}_r,\omega,\boldsymbol{x}_s)$ does not require any renormalization. As in the case of the acoustic medium, the construction of internal multiples is a two-step process: (i) construct the virtual events, and (ii) use (29) to predict internal multiples. The whole process can be wrapped out in a single equation as follows:

$$-\tau^{IM}_{ij}(x_{\mathrm{r}},\omega,x_{\mathrm{s}}) = \overline{a}^{-1}_{ml}(\omega)\overline{h}_{nk}(\omega)\iint_{x,x'\in\partial D_0} n_n(x)n_m(x)\tilde{\tilde{\overline{\tau}}}_{ij}(x_{\mathrm{r}},\omega,x) \tag{30}$$
$$\times\,\tilde{\upsilon}^{b)}_{l}(x,\omega,x')\,\tilde{\upsilon}^{b)}_{k}(x',\omega,x_{\mathrm{s}})dSdS'.$$

Notice that (28) is mathematically equivalent to the expression of virtual events given by Ikelle et al. (in press). However, the field $\tilde{P}(\boldsymbol{x}',\omega,\boldsymbol{x})$ is different from the one used by Ikelle et al. (in press). This difference is actually very important because some choices of $\tilde{P}(\boldsymbol{x}',\omega,\boldsymbol{x})$, like the one we have made here, allow us to avoid the tricky numerical operation of renormalization discussed in Ikelle et al. (in press). For more concreteness, we discuss in the next paragraphs these differences and their implications for the prediction of internal multiples. This discussion will be based on analytical expressions of

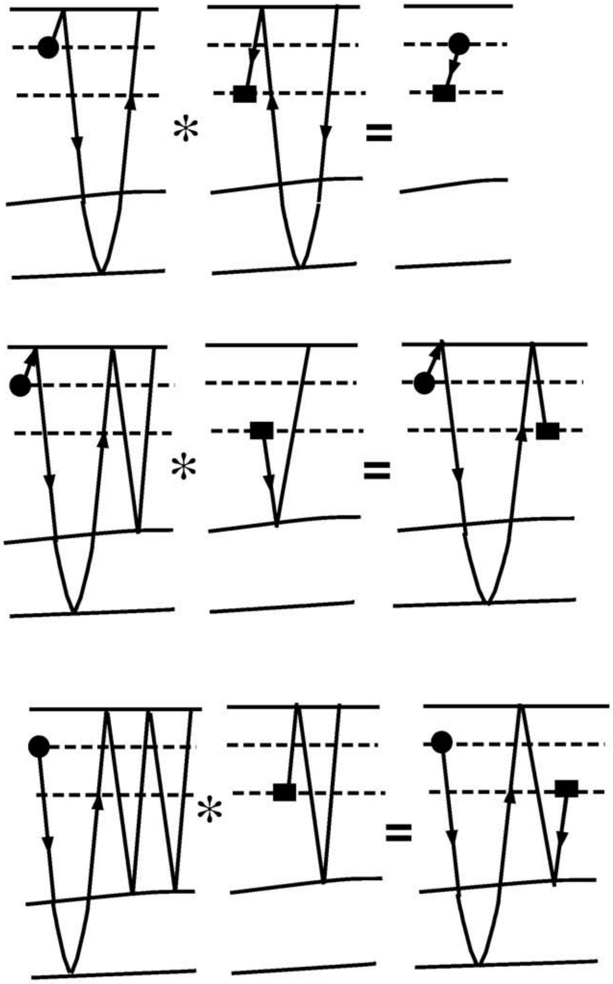

Fig. 3. An illustration of the construction of free-surface multiples, primaries, and internal multiples using the correlation-type Kirchhoff integral. The symbol * denotes the multidimensional crosscorrelation operations.

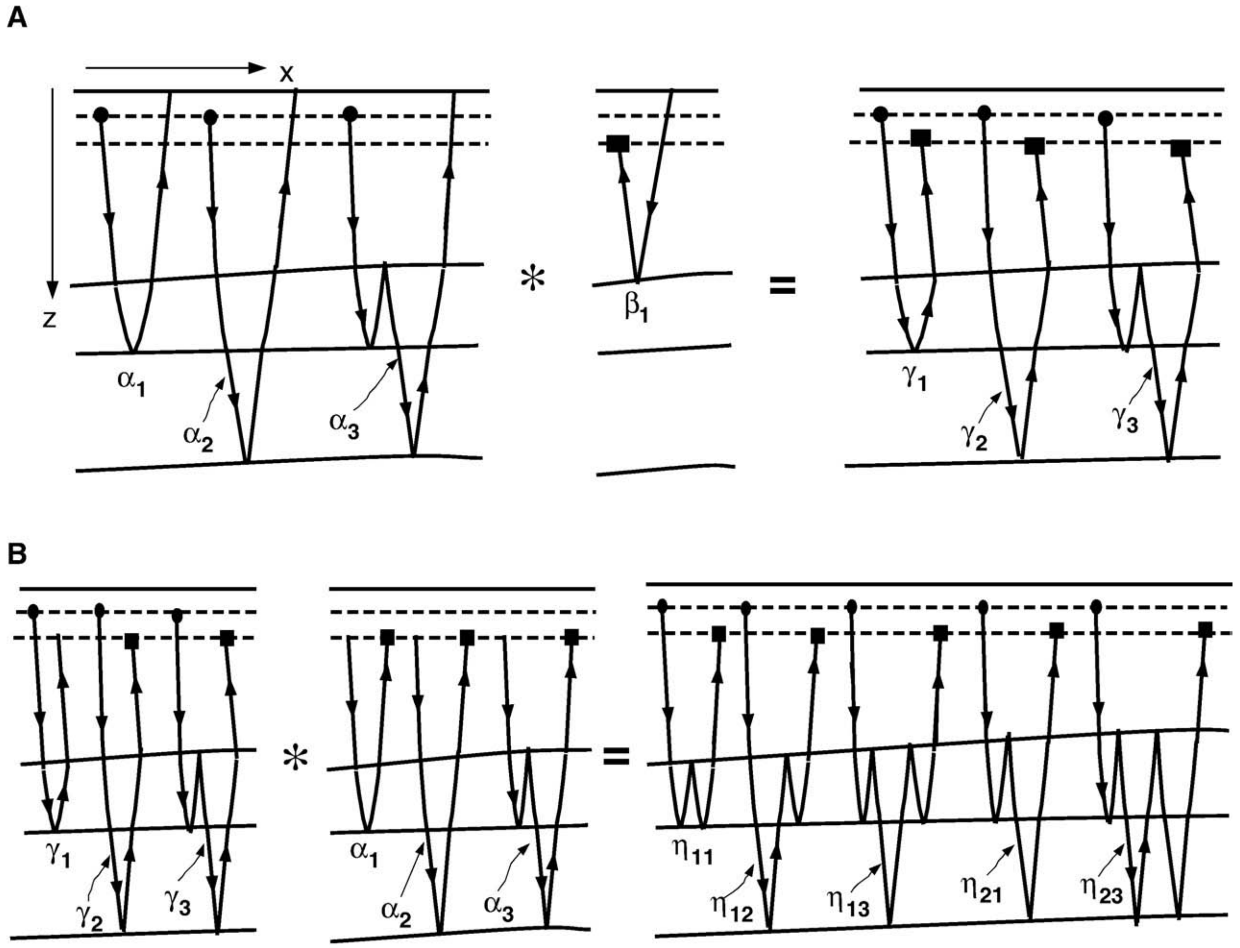

Fig. 4. An illustration with scattering diagrams of the two-step process for generating internal multiples. The first step generates virtual events which are then used in step 2 to generate internal multiples. Notice that the data in the first step data are divided into parts which do not intersect. The early part contains only primaries, and the latter part contained primaries and internal multiples.

internal multiples at the zero offset for a 1D model. The 1D model here consists of two layers sandwiched between two half-spaces, as depicted in Fig. 5.

Because we are dealing with a 1D case at the zero offset, we can analytically compute the internal multiples. Let us start by introducing the downgoing and upgoing propagators with respect to the ith interface, as depicted in Fig. 5. We denote the downgoing propagator by p_i and the upgoing propagator by p_i. In other words, the upgoing propagator is the complex conjugate of the downgoing propagator in the F–X domain. The expression of p_i is

$$p_i = \gamma_i \exp\{-i\omega t_i\}, \tag{31}$$

where γ_i describes the amplitude decay from the source to the ith interface when dealing with the first interface and from the $(i-1)$th interface to the ith interface when dealing with the other interfaces. Similarly, t_i describes the one-way traveltime from the source to the ith interface when dealing with the first interface and from the $(i-1)$th interface to the ith interface when dealing with the other interfaces. We also introduce the reflection and transmission coefficients at the ith interface. So we denote by $R_{i,i+1}$ the reflection coefficient at the ith interface with an upcoming incident wave at the ith interface, and by $R_{i,i-1}$ the reflection coefficient at the ith interface with an downcoming incident wave at the ith interface, again as depicted in Fig. 5. Notice that for $R_{i,i-1}$, i varies from 2 to 4 only. Similarly, denote by $T_{i,i+1}$ the transmission coefficient at the ith interface with an upcoming incident wave at the ith interface, and by $T_{i,i-1}$ the

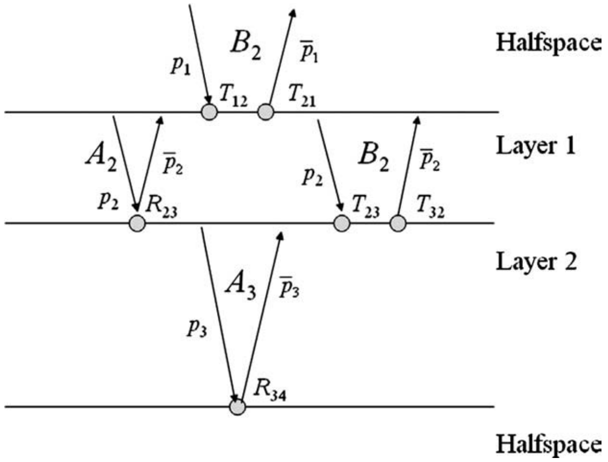

Fig. 5. Definitions of the terms used in Eqs. (31) through (40). p_i denotes the downgoing propagator with respect to ith interface, and its complex conjugate $\bar{p}_i$ denotes the upgoing propagators. $R_{i,i+1}$ is the reflection coefficient at the ith interface with an upcoming incident wave at the ith interface, and $R_{i,i-1}$ is the reflection coefficient at the ith interface with a downcoming incident wave at the ith interface. $T_{i,i+1}$ is the transmission coefficient at the ith interface with an upcoming incident wave at the ith interface, and $T_{i,i-1}$ is the transmission coefficient at the ith interface with a downcoming incident wave at the ith interface. The terms A_i and B_i defined in (34) and (33), respectively, describe the two-way propagations in a given layer or half-space.

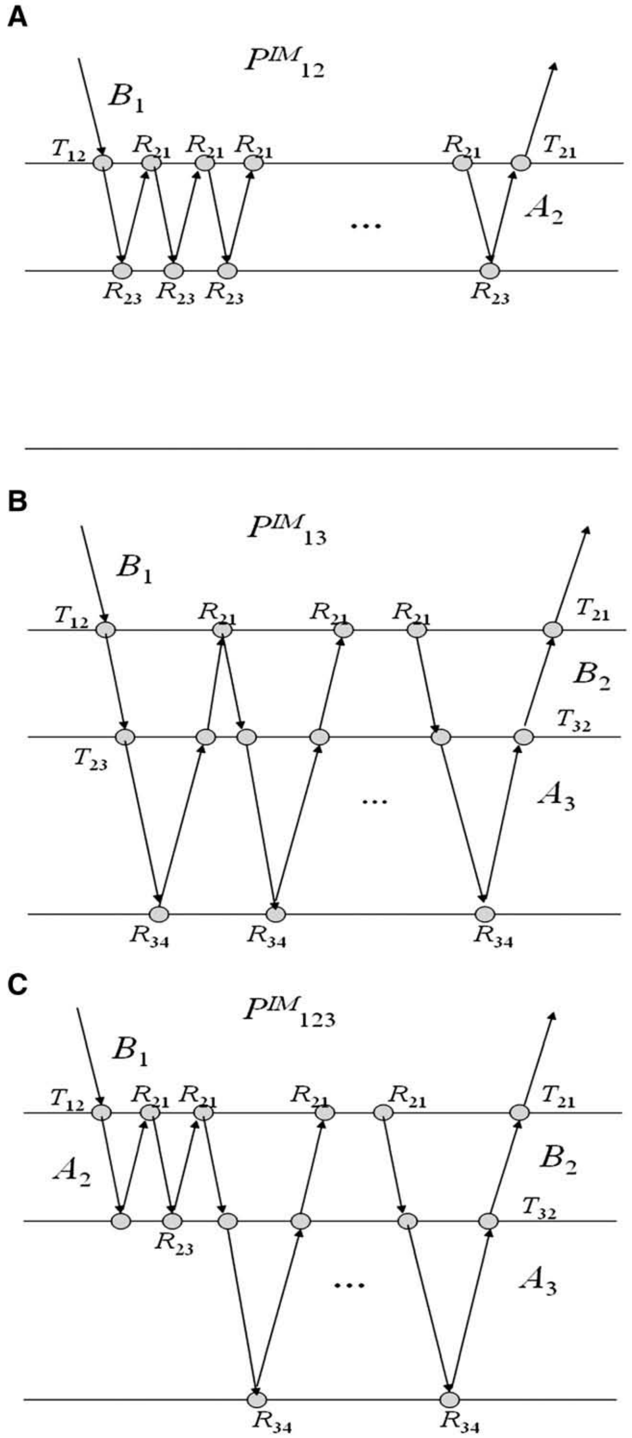

Fig. 6. An illustration of different types of internal multiples observed in a 1D model. The 1D model here consists of two layers sandwiched between two half-spaces. P_{12}^{IM} denotes the multiples which bounce at the first and second interfaces only, P_{13}^{IM} denotes the multiples which bounce at the first and third interfaces only, and P_{123}^{IM} denotes the multiples which bounce at all three interfaces.

transmission coefficient at the *i*th interface with an downcoming incident wave at the *i*th interface, again as depicted in Fig. 5. Notice that for $T_{i,i-1}$, i varies from 2 to 3 only. Using these definitions, we derive the following expression of internal multiples:

$$p^{IM} = B_1A_2\sum_{n=1}(R_{21}A_2)^n + B_1B_2A_3\sum_{m=1}(R_{21}B_2A_3)^m + B_1A_2R_{21}\sum_{n=1,m=1}\frac{(n+m+1)!}{(n+1)!m!}(R_{21}A_2)^n(R_{21}B_2A_3)^m, \tag{32}$$

where

$$B_i = p_iT_{i,i+1}T_{i+1,i}\overline{p}_i \tag{33}$$

and

$$A_i = p_iR_{ii+1}\overline{p}_i. \tag{34}$$

Notice that the first term (which is denoted P_{12}^{IM} in Fig. 6) on the right-hand side of (32) corresponds to the multiples which bounce at the first and second interfaces only (see Fig. 6a), the second term (which is denoted P_{13}^{IM} in Fig. 6) corresponds to multiples which bounce at the first and third interfaces only (see Fig. 6b), and the third term (which is denoted P_{123}^{IM} in Fig. 6) corresponds to multiple bounces at all the three interfaces (see Fig. 6c).

Let us now turn the internal multiple that can be predicted by using the theory described here. By using the notations introduced in (31)–(34), the field of internal multiples, as defined in (32), becomes

$$p^{IM} = \alpha_1B_1A_2\sum_{n=1}(R_{21}A_2)^n + \alpha_2B_1B_2A_3\sum_{m=1}(R_{21}B_2A_3)^m + \alpha_1\alpha_2B_1A_2R_{21}\sum_{n=1,m=1}\frac{(n+m+1)!}{(n+1)!m!}(R_{21}A_2)^n(R_{21}B_2A_3)^m. \tag{35}$$

Notice that our expression differs from the exact expression of the internal multiple in (32) by the coefficients α_1 and α_2. These differences are generally compensated at the subtraction stage of the multiple attenuation.

Let us turn to the prediction of internal multiples without the correction introduced in this paper. We need the vertical component of the particle velocity for this prediction. Using the well-known relationship between the pressure and the vertical component of the particle velocity (Ikelle and Amundsen, 2005), we can also express the vertical component of the particle velocity as follows:

$$\upsilon = \tilde{A}_1 + \tilde{B}_1\sum_{m=0}\left(R_{2,1}\tilde{B}_2\tilde{A}_3\right)^m \times \left[\tilde{A}_2\sum_{n=0}\frac{(n+m+1)!}{(n+1)!m!}\left(R_{2,1}\tilde{A}_2\right)^n + \tilde{B}_2\tilde{A}_3\right] + \ldots, \tag{36}$$

where

$$\tilde{B}_i = p_i'T_{i,i+1}T_{i+1,i}\overline{p}_i' \tag{37}$$

$$\tilde{A}_i = p_i'R_{i,i+1}\overline{p}_i'. \tag{38}$$

and

$$p_i' = \gamma_i'\exp\{-i\omega t_i\}. \tag{39}$$

The quantity γ_i' describes the amplitude decay. It also captures the difference in polarity and scale between the pressure and the particle velocity. By combining (32) and (36), we obtain the following expression of the field of predicted internal multiples:

$$p^{IM} = \alpha_1\tilde{B}_1\tilde{A}_2R_{21}\tilde{A}_2\sum_{n=0,m=0}\left(R_{21}\tilde{A}_2\right)^{n+m} + \alpha_1\tilde{B}_1\tilde{B}_2\tilde{A}_3R_{21}\tilde{B}_2\tilde{A}_3\sum_{n=0,m=0}\left(R_{21}\tilde{B}_2\tilde{A}_3\right)^{n+m} + 2\alpha_1\tilde{B}_1R_{21}^{-1}\sum_{n=0,m=0}\sum_{k=1,l=1}\frac{(n+m+1)!}{(n+1)!m!}\frac{(k+l+1)!}{(k+1)!l!}\left(R_{21}\tilde{A}_2\right)^{n+k}\left(R_{21}\tilde{B}_2\tilde{A}_3\right)^{m+l} \tag{40}$$

The derivation of this expression is given in appendix A. Notice that this expression is not reducible to the exact expression of the internal multiples in (32), even with a smart subtraction technique, because each set of internal multiples requires different correcting terms.

4. Conclusions

To conclude, we have extended the Kirchhoff integral formulation of multiple attenuation, and free-surface multiples, as well as internal multiples, to the elastic case. For a convolution-type representation theorem, we showed that each component of the stress tensor field can be demultipled separately as long as all of the components of the particle-velocity field are available. We obtained similar series for the components of the particle-velocity field.

Using the correlation-type representation theorem, we proposed a way of estimating actual time-retarded data without free-surface multiples in the form of the Kirchhoff series. We also showed that when this time-retarded field is convoluted with the actual data, we can predict internal multiples without the cumbersome renormalization operation encountered in previous derivations.

Appendix A

Our objective in this appendix is to describe how we obtained the results in (32), (35), and (40).

There are three primaries associated with the 1D model in Fig. 5. Using the notations in (31)–(34), the first primary to arrive at the receiver location can be defined as A_1, the second primary as B_1A_2, and the third primary as $B_1B_2A_3$. The internal multiples can be constructed as the product of these primaries. That is,

$$\begin{aligned} p^{IM} = {} & B_1A_2 \sum_{n=1} (R_{2,1}A_2)^n + B_1B_2A_3 \sum_{m=1} (R_{2,1}B_2A_3)^m \\ & + B_1A_2R_{2,1} \sum_{n=1,m=1} \frac{(n+m+1)!}{(n+1)!m!} (R_{2,1}A_2)^n (R_{2,1}B_2A_3)^m, \end{aligned} \tag{41}$$

which is the expression of internal multiples in (32).

By summing the primary and internal-multiple events, we obtain the total field; i.e.,

$$P = A_1 + B_1 \sum_{m=0} (R_{2,1}B_2A_3)^m \left[A_2 \frac{(n+m+1)!}{(n+1)!m!} \sum_{n=0} (R_{2,1}A_2)^n + B_2A_3 \right] + \dots . \tag{42}$$

We are now in the position of computing the field of virtuals. The pressure field of virtual events associated with the first subsurface reflector is defined as

$$P_l^V = P_1^{-1} \upsilon_{2+}, \tag{43}$$

where P_1^{-1} is the inverted pressure field for the first primary and υ_{2+} is the field of the vertical component of the particle velocity for all of the seismic events for which traveltimes are greater than that of the first primary. Then the internal multiples associated with the first subsurface reflector can be calculated as follows:

$$P_1^{IM} = P_1^V \upsilon_{2+}. \tag{44}$$

By definition, the inverted pressure field of the first primary is

$$P_1^{-1} = A_1^{-1}, \tag{45}$$

and the field υ_{2+} is

$$\begin{aligned} \upsilon_{2+} = {} & \tilde{B}_1 \tilde{A}_2 + \tilde{B}_1 \tilde{B}_2 + \tilde{B}_1 \tilde{A}_2 \sum_{n=1} \left(R_{21} \tilde{A}_2\right)^n + \tilde{B}_1 \tilde{B}_2 \tilde{A}_3 \sum_{m=1} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^m \\ & + \tilde{B}_1 \tilde{A}_2 R_{21} \sum_{n=1,m=1} \frac{(n+m+1)!}{(n+1)!m!} \left(R_{21} \tilde{A}_2\right)^n \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^m. \end{aligned} \tag{46}$$

By substituting (45) and (46) into (43), we obtain the field of virtual events associated with the first reflector:

$$P_1^V = A_1^{-1} \tilde{B}_1 \sum_{m=0} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^m \times \left[\tilde{A}_2 \frac{(n+m+1)!}{(n+1)!m!} \sum_{n=0} \left(R_{21} \tilde{A}_2\right)^n + \tilde{B}_2 \tilde{A}_3 \right]. \tag{47}$$

Similarly, by substituting (47) and (46) into (44), we obtain the field of internal multiples

$$\begin{aligned} P_1^{IM} = {} & A_1^{-1} \tilde{B}_1 \sum_{m=0} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^m \left[\tilde{A}_2 \frac{(n+m+1)!}{(n+1)!m!} \sum_{n=0} \left(R_{21} \tilde{A}_2\right)^n + \tilde{B}_2 \tilde{A}_3 \right] \\ & \times \tilde{B}_1 \sum_{l=0} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^l \left[\tilde{A}_2 \frac{(k+l+1)!}{(k+1)!l!} \sum_{k=0} \left(R_{21} \tilde{A}_2\right)^k + \tilde{B}_2 \tilde{A}_3 \right]. \end{aligned} \tag{48}$$

Note that (48) can be rewritten

$$\begin{aligned} P_1^{IM} = {} & \alpha_1 \tilde{B}_1 \tilde{A}_2 R_{21} \tilde{A}_2 \sum_{n=0,m=0} \left(R_{21} \tilde{A}_2\right)^{n+m} \\ & + \alpha_1 \tilde{B}_1 \tilde{B}_2 \tilde{A}_3 R_{21} \tilde{B}_2 \tilde{A}_3 \sum_{n=0,m=0} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^{n+m} \\ & + 2\alpha_1 \tilde{B}_1 R_{21}^{-1} \sum_{n=0,m=0} \sum_{k=1,l=1} \frac{(n+m+1)!}{(n+1)!m!} \frac{(k+l+1)!}{(k+l+1)!l!} \left(R_{21} \tilde{A}_2\right)^{n+k} \left(R_{21} \tilde{B}_2 \tilde{A}_3\right)^{m+l} \end{aligned} \tag{49}$$

We have used the fact that

$$A_1^{-1} \tilde{B}_1 = \left(\frac{\gamma_1}{\gamma_1'}\right)^2 \times \frac{T_{12}T_{21}}{R_{12}} = \alpha_1 R_{21} \tag{50}$$

in the derivation of (49). So (49) is the second formula that we wanted to derive in this appendix; that is, the field of internal multiples without the correction in (40).

Let us now derive expression (35). We have discovered, that the field

$$\tilde{P}_1^{-1} = P_1^{-1}(1 - \upsilon_{2+}) \tag{51}$$

which is used instead of P_1^{-1}, allows us to reduce Eq. (40) to the form that is more closely related to that in Eq. (32). Note that we have here assumed that the source signature $a(\omega)$ is equal to 1, implying that υ_{2+} in (51) is dimensionless. Now, by substituting (51) into (43), we obtain the Eq. (35); that is,

$$\begin{aligned} P_1^{IM} = {} & \alpha_1 B_1 A_2 \sum_{n=1} (R_{21}A_2)^n + \alpha_2 B_1 B_2 A_3 \sum_{m=1} (R_{21}B_2A_3)^n + \alpha_1 \alpha_2 B_1 A_2 R_{21} B_2 A_3 \\ & \times \sum_{m=1,n=1} \frac{(n+m+1)!}{(n+1)!m!} (R_{21}A_2)^n (R_{21}B_2A_3)^l, \end{aligned} \tag{52}$$

with

$$\alpha_1 = \left(\frac{\gamma_1}{\gamma_1'}\right)^2 \frac{T_{12}T_{21}}{R_{12}R_{21}}, \tag{53}$$

$$\alpha_2 = \left(\frac{\gamma_1\gamma_2}{\gamma_1'\gamma_2'}\right)^2 \frac{T_{12}T_{21}}{R_{12}R_{21}}. \tag{54}$$

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