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Formulation of the linearized forward problems for multicomponent OBS data in a water/solid configuration using the reciprocity theorem

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SUMMARY

As is the case with other geophysical data, the interpretation of multicomponent data from the new ocean bottom seismic (OBS) acquisition system requires a solution to the forward problem of predicting seismograms. Born approximation (a single-scattering approximation) is one of the classical tools used for solving forward problems. The adaptation of this solution to OBS data requires (1) the explicit introduction of the boundary conditions at a water/solid interface, (2) the use of the reciprocity theorem in a fluid/solid configuration and (3) the use of P- and S-wave potentials. We present a formulation of the linearized forward problem for predicting multicomponent OBS data using Born approximation and the reciprocity theorem. The results of our formulation are valid for pressure data (hydrophone data) as well as particle velocity data (geophone data). Specific cases where particle velocity data are cast into P- and S-wave potentials are also treated. For these scenarios, the background medium and the scatterer are considered arbitrarily anisotropic and heterogeneous.

Key words: Born approximation, forward problems, multicomponent data, ocean bottom seismic (OBS).

1 INTRODUCTION

Multicomponent ocean bottom seismic (OBS) data, where pressure data (hydrophone data) are recorded just above the seafloor and particle velocity data (geophone data) just below the seafloor, can significantly improve the quality and quantity of information retrievable from seismic data. The expectations behind OBS surveys are many; they range from seismic imaging to fluid prediction and even to fluid movement. Converting these expectations into reality is a difficult but exciting challenge for petroleum seismologists.

As with other geophysical data, the problem of interpreting multicomponent OBS data can be narrowed down to the problem of obtaining an earth model that will best predict actual observed seismograms. This raises two questions: (1) given an earth model, how do we solve the forward problem of predicting seismograms, and (2) how do we solve the inverse problem of obtaining an optimum earth model? In this paper, we focus on the first question.

Born approximation (a single-scattering approximation) is one of the classic tools used for solving forward problems. Despite being limited to small variations in medium parameters, the Born-based solution of the forward problem provides us with a physical insight into the interaction between parameters, which describe the subsurface, and seismograms, which are very useful for interpreting data. Furthermore, its closed form makes Born approximation very attractive for constructing inversion algorithms (e.g. Snieder 1990; Ikelle 1995). However, the adaptation of Born-based solutions to the forward problem of multicomponent OBS data requires (1) the explicit introduction of the boundary conditions at a water/solid interface, (2) the use of the reciprocity theorem for a fluid/solid configuration and (3) the use of P- and S-wave potentials.

Here we give a solution to the forward problem of multicomponent OBS data within the limits of the Born approximation. We will use existing results of the reciprocity theorem for a fluid/solid configuration (e.g. Gangi 1970; Bleistein 1984; de Hoop 1990; Fokkema & van den Berg 1993; Wapenaar & Grimbergen 1996; Thomsen 1998; Wapenaar & Fokkema 1999; Amundsen 1999; Amundsen & Ikelle 1999).

We will begin by introducing the notation convention and some basic elastodynamic equations. In Section 3, we will derive the general scattering volume integral equations for a general case (i.e. one without the weak scattering approximation). In Section 4, we use these integral equations to deduce the linearized forward problem solution under the weak scattering approximation.

2 INTRODUCTORY DEFINITIONS

Consider a 3-D model of the earth consisting of an inhomogeneous solid half-space overlain by a homogeneous fluid (water) layer as described in Fig. 1. We will divide this model into two subdomains: (1) the subdomain occupied by a solid, denoted D_{s}, and (2) the subdomain occupied by water, denoted D_{f}. An interface between the two subdomains will be denoted A. The forward scattering problem will be formulated by introducing an inclusion, Ω_{s}, such that in its absence, the rest of the model represents the background medium (also known as the reference medium). The inclusion Ω_{s} will be called the medium perturbation. We will assume that the fluid occupying the subdomain D_{f} is homogeneous, and that the solid occupying the subdomain D_{s}, including Ω_{s}, is arbitrarily inhomogeneous and anisotropic in its elastic behaviour. Our goal in this paper is to derive a formula for predicting seismograms that correspond to such a model of the earth. In this section we introduce our notation and recall some of the basic elastodynamic equations needed for this formulation.

2.1 Notation conventions

Position in this configuration is specified by the coordinates $\{x, y, z\} = \{x_1, x_2, x_3\}$ with respect to a fixed Cartesian reference frame with the origin O and three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ each of unit length; $\mathbf{i}_3$ points vertically downwards. To accommodate anisotropy, the subscript notation for vectors and tensors and the Einstein summation convention are adopted. Lower-case italic subscripts are employed for this purpose (e.g. v_i, τ_{ij}); they are assigned the values 1, 2 and 3 unless specified otherwise. Subscripts s and r indicate source and receiver, respectively. Bold symbols (e.g. $\mathbf{v}$, $\boldsymbol{\tau}$) are used to indicate vectors and tensors. Partial differentiation with respect to x_m is denoted by ∂_m; ∂_t is reserved for partial differentiation with respect to time, t.

2.2 Basic equations for elastodynamic wave motion

In the subdomain D_{f} (occupied by water), the acoustic wavefield can be characterized by the acoustic pressure, denoted here by $\sigma = \sigma(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ and the water particle velocity, denoted by $w_r = w_r(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ for a shotpoint located at $\mathbf{x}_{\mathrm{s}}$ in either D_{f} or D_{s} and for a point $\mathbf{x}$ in the fluid. These quantities satisfy the equations

$$\partial_k \sigma + \rho_{\mathrm{f}} \partial_t w_k = 0\,, \tag{1}$$

$$\partial_r w_r + \kappa \partial_t \sigma = q\,, \tag{2}$$

with

$$q = q(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}}) = a(t)\delta(\mathbf{x} - \mathbf{x}_{\mathrm{s}})\,, \tag{3}$$

in which $q = q(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ is the water volume source density injection rate, ρ_{f} is the water's volume density of mass and κ is the water compressibility. ρ_{f} and κ are constant as we have assumed that the water layer occupying D_{f} is a homogeneous medium.

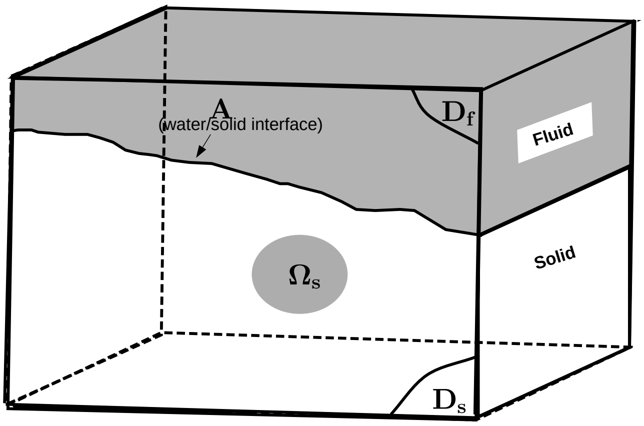

Figure 1. Water/solid configuration with an inclusion in the solid.

In the subdomain D_{s} (occupied by a solid), the elastic wavefield can be characterized by the solid stress, denoted here by $\tau_{pq} = \tau_{pq}(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$, and the solid particle velocity, denoted by $v_r = v_r(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ for a shotpoint located at $\mathbf{x}_{\mathrm{s}}$ in either D_{f} or D_{s} and for receiver point $\mathbf{x}$ in the solid. These quantities satisfy the equations

$$-\Delta_{kmpq} \partial_m \tau_{pq} + \rho_{\mathrm{s}} \partial_t v_k = 0\,, \tag{4}$$

$$\Delta_{ijmr} \partial_m v_r - s_{ijpq} \partial_t \tau_{pq} = 0\,, \tag{5}$$

where

$$\Delta_{ijpq} = \frac{1}{2}\,(\delta_{ip}\,\delta_{jq} + \delta_{iq}\,\delta_{jp}) \tag{6}$$

and in which $\rho_{\mathrm{s}} = \rho_{\mathrm{s}}(\mathbf{x})$ is the solid volume density of mass and $s_{ijpq} = s_{ijpq}(\mathbf{x})$ is the compliance. Δ_{ijpq} is the symmetrical unit tensor of rank four that is characteristic for elastodynamics. In eq. (5), δ_{ip} is the symmetrical unit tensor of rank two (Kronecker tensor): $\delta_{ip} = \{0, 1\}$ for $\{i \neq p, i = p\}$. Alternatively, the stiffness tensor c_{ijpq} can be used to characterize the solid instead of elastic compliances s_{ijpq}; the stiffness tensor c_{ijpq} is the inverse of the tensor of compliances:

$$s_{ijrs}\, c_{rspq} = \Delta_{ijpq}\,. \tag{7}$$

Wave equations in (1)–(4) suggest, in theory, that while several quantities can be used to measure wave motion $\{\sigma, w_r, \tau_{pq}, v_k\}$, only two of these quantities are recorded in this OBS experiment:

(1) the pressure $\sigma(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ at a point $\mathbf{x}$ in the fluid just above the water/solid interface and for a shotpoint in the water;

(2) the particle velocity $v_r(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})$ at a point $\mathbf{x}$ in the solid just below the water/solid interface for the same shotpoint in the water.

The problem we will try to solve in the next sections is that of predicting the wavefields σ and v_r for a given model $\{\rho_{\mathrm{f}}, \kappa, \rho_{\mathrm{s}}, \mathbf{s}\}$. To make the subsequent notations more compact, we introduce the model set

$$\mathbf{m} = \begin{pmatrix} \rho_{\mathrm{f}} \\ \kappa \\ \rho_{\mathrm{s}} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \rho_{\mathrm{f}} \\ \kappa \\ \rho_{\mathrm{s}}^{(0)} \\ \mathbf{s}^{(0)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \rho_{\mathrm{s}}^{(1)} \\ \mathbf{s}^{(1)} \end{pmatrix} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}\,. \tag{8}$$

Note that, in anticipation of the Born approximation, which we will use later on, we have decomposed the medium parameter corresponding to the subdomain D_{s} into a reference and a perturbation, i.e.

$$\rho_s = \rho_s^{(0)} + \rho_s^{(1)}\,, \tag{9}$$

$$c_{ijpq} = c_{ijpq}^{(0)} + c_{ijpq}^{(1)}\,, \tag{10}$$

$$s_{ijpq} = s_{ijpq}^{(0)} + s_{ijpq}^{(1)}\,, \tag{11}$$

with

$$s_{ijrs}^{(0)}\, c_{rspq}^{(0)} = \Delta_{ijpq}\,, \tag{12}$$

where $\rho_s^{(0)}$, $c_{ijpq}^{(0)}$ and $s_{ijpq}^{(0)}$ correspond to the reference medium and $\rho_s^{(1)}$, $c_{ijpq}^{(1)}$ and $s_{ijpq}^{(1)}$ correspond to the perturbation. By substituting (10) and (11) into (7), using (12), we obtain the following relationship between $c_{ijpq}^{(1)}$ and $s_{ijpq}^{(1)}$:

$$s_{ijrs}^{(0)}\, c_{rspq}^{(1)} + s_{ijrs}^{(1)}\, c_{rspq}^{(0)} + s_{ijrs}^{(1)}\, c_{rspq}^{(1)} = 0\,. \tag{13}$$

For a particular case of weak perturbation, $s_{ijrs}^{(1)}\, c_{rspq}^{(1)}$ can be considered as the second-order approximation and therefore be neglected. Identity (13) becomes

$$s_{ijrs}^{(0)}\, c_{rspq}^{(1)} + s_{ijrs}^{(1)}\, c_{rspq}^{(0)} \cong 0\,. \tag{14}$$

This relationship will be useful in the Born approximation forward problem that we will describe in the following sections.

3 SCATTERING FORWARD PROBLEM: GENERAL CASE

3.1 Setting up the scattering forward problem

In accordance with the standard scattering description, the 'wave motion' $\{\sigma, w_r, \tau_{pq}, v_r\}$ is decomposed into (1) a direct part $\{\sigma^{(0)}, w_r^{(0)}, \tau_{pq}^{(0)}, v_r^{(0)}\}$, which is wave motion corresponding to a reference medium, $\mathbf{m}^{(0)}$, and (2) a scattered part, $\{\sigma^{(1)}, w_r^{(1)}, \tau_{pq}^{(1)}, v_r^{(1)}\}$, which is an additional wavefield due to the presence of the perturbation $\mathbf{m}^{(1)}$. Hence,

$$\{\sigma, w_r\}(\mathbf{x}, t, \mathbf{x}_s) = \{\sigma^{(0)} + \sigma^{(1)}, w_r^{(0)} + w_r^{(1)}\}(\mathbf{x}, t, \mathbf{x}_s)\,, \qquad \mathbf{x} \in D_f(\text{water})\,, \tag{15}$$

$$\{\tau_{pq}, v_r\}(\mathbf{x}, t, \mathbf{x}_s) = \{\tau_{pq}^{(0)} + \tau_{pq}^{(1)}, v_r^{(0)} + v_r^{(1)}\}(\mathbf{x}, t, \mathbf{x}_s)\,, \qquad \mathbf{x} \in D_s(\text{solid})\,, \tag{16}$$

where the direct wave then satisfies the first-order coupled wave equations

$$\begin{cases} \partial_k \sigma^{(0)}(\mathbf{x}, t, \mathbf{x}_s) + \rho_f\, \partial_t w_k^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \\ \partial_r w_r^{(0)}(\mathbf{x}, t, \mathbf{x}_s) + \kappa\, \partial_t \sigma^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = q(\mathbf{x}, t, \mathbf{x}_s) \end{cases} \qquad \mathbf{x} \in D_f(\text{water})\,, \tag{17}$$

$$\begin{cases} -\Delta_{kmpq} \partial_m \tau_{pq}^{(0)}(\mathbf{x}, t, \mathbf{x}_s) + \rho_s^{(0)}(\mathbf{x}) \partial_t v_k^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \\ \Delta_{ijmr} \partial_m v_r^{(0)}(\mathbf{x}, t, \mathbf{x}_s) - s_{ijpq}^{(0)}(\mathbf{x}) \partial_t \tau_{pq}^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \end{cases} \qquad \mathbf{x} \in D_s(\text{solid})\,, \tag{18}$$

with the boundary conditions (on $A(\mathbf{x})$)

$$\begin{cases} \nu_m(\mathbf{x}) w_m^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = \nu_m(\mathbf{x}) v_m^{(0)}(\mathbf{x}, t, \mathbf{x}_s) \\ \sigma^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = -\nu_k(\mathbf{x}) \Delta_{kmpq} \nu_m(\mathbf{x}) \tau_{pq}^{(0)}(\mathbf{x}, t, \mathbf{x}_s) \\ (\delta_{kr} - \nu_k(\mathbf{x}) \nu_r(\mathbf{x})) \Delta_{rmpq} \nu_m(\mathbf{x}) \tau_{pq}^{(0)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \end{cases} \qquad \mathbf{x} \in A(\text{water/solid})\,, \tag{19}$$

where ν_m is the m-component of the local unit along the normal to the water/solid interface, denoted A. If we substitute (15) and (16) into eqs (1)–(4) and if we use the decomposition of the medium into the reference medium and the perturbation in (8), we obtain the following equations for the scattered wavefield:

$$\begin{cases} \partial_k \sigma^{(1)}(\mathbf{x}, t, \mathbf{x}_s) + \rho_f\, \partial_t w_k^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \\ \partial_r w_r^{(1)}(\mathbf{x}, t, \mathbf{x}_s) + \kappa\, \partial_t \sigma^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \end{cases} \qquad \mathbf{x} \in D_f(\text{water})\,, \tag{20}$$

$$\begin{cases} -\Delta_{kmpq} \partial_m \tau_{pq}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) + \rho_s^{(0)}(\mathbf{x}) \partial_t v_k^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = f_k^{(1)}(\mathbf{x}, t, \mathbf{x}_s) \\ \Delta_{ijmr} \partial_m v_r^{(1)}(\mathbf{x}, t, \mathbf{x}_s) - s_{ijpq}^{(0)}(\mathbf{x}) \partial_t \tau_{pq}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = h_{ij}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) \end{cases} \qquad \mathbf{x} \in D_s(\text{solid})\,, \tag{21}$$

where

$$f_k^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = -\rho_s^{(1)}(\mathbf{x})\, \partial_t v_k(\mathbf{x}, t, \mathbf{x}_s)\,, \qquad \mathbf{x} \in \Omega_s(\text{inclusion})\,, \tag{22}$$

$$h_{ij}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = s_{ijpq}^{(1)}(\mathbf{x})\, \partial_t \tau_{pq}(\mathbf{x}, t, \mathbf{x}_s)\,, \qquad \mathbf{x} \in \Omega_s(\text{inclusion})\,. \tag{23}$$

$f_k^{(1)}$ and $h_{ij}^{(1)}$ are the contrasts of volume source densities of force and deformation rate in Ω_s, respectively, that generate the scattered waves.

The boundary conditions for the scattered wavefield are

$$\begin{cases} \nu_m(\mathbf{x}) w_m^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = \nu_m(\mathbf{x}) v_m^{(1)}(\mathbf{x}, t, \mathbf{x}_s) \\ \sigma^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = -\nu_k(\mathbf{x}) \Delta_{kmpq} \nu_m(\mathbf{x}) \tau_{pq}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) \\ (\delta_{kr} - \nu_k(\mathbf{x}) \nu_r(\mathbf{x})) \Delta_{rmpq} \nu_m(\mathbf{x}) \tau_{pq}^{(1)}(\mathbf{x}, t, \mathbf{x}_s) = 0 \end{cases} \qquad \mathbf{x} \in A(\text{water/solid})\,. \tag{24}$$

3.2 Introduction of Green's tensors

Our goal in this paper is to find a relation that will allow us to predict multicomponent OBS seismic data. In other words, we want to estimate the scattered particle velocity field $v_r^{(1)}$ and/or the scattered acoustic pressure field $\sigma^{(1)}$ as a function of $\rho_s^{(1)}$ and $s_{ijpq}^{(1)}$ by solving eqs (20) and (21). For a given system of sources, $\{0, 0, f_k^{(1)}, h_{ij}^{(1)}\}$, the wave propagation in eqs (20) and (21) takes place in the reference medium. To solve this system of equations, we will need the Green's tensors associated with the reference medium $\mathbf{m}^{(0)}$. Because the 'wave motion' is described here by four quantities $\{\sigma, w_r, \tau_{pq}, v_k\}$ and four source types $\{F_k, q, f_k, h_{ij}\}$ (F_k is the water volume source density of force, q is the water volume source density of injection rate, f_k is the solid volume source density of force and h_{ij} is the solid volume source density of rate of deformation), the solution to this problem requires 16 Green's tensors (see Table 1).

Since $\mathbf{m}^{(0)}$ is assumed to be independent of time, the Green's tensors in Table 1 are invariant with respect to time translation,

Table 1. Elastodynamic Green's tensors.

Observed quantity	Water injection rate at $(\mathbf{x}_s, t')$	Water density of force at $(\mathbf{x}_s, t')$	Solid deformation rate at $(\mathbf{x}_s, t')$	Solid density of force at $(\mathbf{x}_s, t')$
Water particle velocity at $(\mathbf{x}, t)$	$G_k^{w;q}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rk}^{w;F}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rij}^{w;h}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rk}^{w;f}(\mathbf{x}, t; \mathbf{x}_s, t')$
Acoustic pressure at $(\mathbf{x}, t)$	$G^{\sigma;q}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_k^{\sigma;F}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{ij}^{\sigma;h}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_k^{\sigma;f}(\mathbf{x}, t; \mathbf{x}_s, t')$
Solid particle velocity at $(\mathbf{x}, t)$	$G_r^{v;q}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rk}^{v;F}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rij}^{v;h}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rk}^{v;f}(\mathbf{x}, t; \mathbf{x}_s, t')$
Stress at $(\mathbf{x}, t)$	$G_{rk}^{\tau;q}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rki}^{\tau;F}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rkij}^{\tau;h}(\mathbf{x}, t; \mathbf{x}_s, t')$	$G_{rki}^{\tau;f}(\mathbf{x}, t; \mathbf{x}_s, t')$

e.g.

$$G_{rk}^{v;f}(\mathbf{x}, t; \mathbf{x}_s, t') = G_{rk}^{v;f}(\mathbf{x}, t-t'; \mathbf{x}_s, 0) = G_{rk}^{v;f}(\mathbf{x}, 0; \mathbf{x}_s, t'-t)\,. \quad (25)$$

The same property holds true for the other Green's tensors in Table 1. To shorten the notation, we replace $G_{rk}^{v;f}(\mathbf{x}, t; \mathbf{x}_s, 0)$ with $G_{rk}^{v;f}(\mathbf{x}, t; \mathbf{x}_s)$.

The 16 Green's tensors in Table 1 are related via the wave equations and the reciprocity theorems such that we actually only need three of them (see Appendix A for more details). We can regroup them according to source and observation points as follows.

(1) Four Green's tensors describe the case where source and observation points are in water (D_f): $G^{\sigma;q}$, $G_k^{\sigma;F}$, $G_k^{w;q}$ and $G_{rk}^{w;F}$. These four Green's tensors are related such that we only need to determine one, for instance $G^{\sigma;q}$ (see Table 2). It is governed by the acoustic wave equation,

$$\frac{1}{v_P^2}\,\partial_t^2 G^{\sigma;q}(\mathbf{x}, t; \mathbf{x}') - \partial_m^2 G^{\sigma;q}(\mathbf{x}, t; \mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,\partial_t[\delta(t)]\,,$$

$$\mathbf{x} \in D_f(\text{water})\,, \quad (26)$$

with the following boundary condition at the water/solid interface:

$$-\frac{1}{\rho_f}\,v_m(\mathbf{x})\partial_m G^{\sigma;q}(\mathbf{x}, t; \mathbf{x}') = v_m(\mathbf{x})\partial_t G_m^{v;q}(\mathbf{x}, t; \mathbf{x}')\,,$$

$$\mathbf{x} \in A(\text{water/solid})\,, \quad (27)$$

where $G_r^{v;q}$ satisfies the following elastic wave equation:

$$\rho_s^{(0)}(\mathbf{x})\,\partial_t^2 G_k^{v;q}(\mathbf{x}, t; \mathbf{x}') - \partial_m(C_{kmij}^{(0)}(\mathbf{x})\,\partial_i)G_j^{v;q}(\mathbf{x}, t; \mathbf{x}') = 0\,,$$

$$\mathbf{x} \in D_s(\text{solid})\,, \quad (28)$$

and

$$v_P^2 = (\rho_f \kappa)^{-1}\,.$$

Table 2. Relationships amongst the 16 elastodynamic Green's tensors introduced in Table 1. The proofs of these relationships are given in Appendix A.

Relationships	Basis of the relation
$G_k^{\sigma;F}(\mathbf{x}, t, \mathbf{x}') = G_k^{w;q}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$\partial_r G_r^{w;q}(\mathbf{x}, t, \mathbf{x}') = -\kappa\,\partial_t G^{\sigma;q}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$\partial_r G_{rk}^{w;F}(\mathbf{x}, t, \mathbf{x}') = -\kappa\,\partial_t G_k^{\sigma;F}(\mathbf{x}, t, \mathbf{x}')$ if $\mathbf{x} \neq \mathbf{x}'$	'Wave motion'
$G_k^{\sigma;f}(\mathbf{x}, t, \mathbf{x}') = G_k^{v;q}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$G_{pq}^{\sigma;h}(\mathbf{x}, t, \mathbf{x}') = G_{pq}^{\tau;q}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$G_{rk}^{w;f}(\mathbf{x}, t, \mathbf{x}') = G_{rk}^{v;F}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$G_{rij}^{w;h}(\mathbf{x}, t, \mathbf{x}') = G_{ijr}^{\tau;F}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$\partial_t G_{pq}^{\tau;q}(\mathbf{x}, t, \mathbf{x}') = c_{pqij}^{(0)}(\mathbf{x})\partial_i G_j^{v;q}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$\partial_t G_{pqk}^{\tau;F}(\mathbf{x}, t, \mathbf{x}') = c_{pqij}^{(0)}(\mathbf{x})\partial_i G_{ik}^{v;F}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$\partial_r G_{rk}^{w;f}(\mathbf{x}, t, \mathbf{x}') = -\kappa\,\partial_t G_k^{\sigma;f}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$\partial_r G_{rij}^{w;h}(\mathbf{x}, t, \mathbf{x}') = -\kappa\,\partial_t G_{ij}^{\sigma;h}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$G_{rij}^{v;h}(\mathbf{x}, t, \mathbf{x}') = G_{ijr}^{\tau;f}(\mathbf{x}', t, \mathbf{x})$	Reciprocity theorem
$\partial_t G_{pqk}^{\tau;f}(\mathbf{x}, t, \mathbf{x}') = c_{pqij}^{(0)}(\mathbf{x})\partial_i G_{jk}^{v;f}(\mathbf{x}, t, \mathbf{x}')$	'Wave motion'
$\partial_t G_{pqij}^{\tau;h}(\mathbf{x}, t, \mathbf{x}') = c_{pqmr}^{(0)}(\mathbf{x})\partial_m G_{rij}^{v;f}(\mathbf{x}, t, \mathbf{x}')$ if $\mathbf{x} \neq \mathbf{x}'$	'Wave motion'

(2) Eight Green's tensors describe the case where the source point is in the water and the observation point is in the solid or vice versa: $G_k^{\sigma;f}$, $G_{ij}^{\sigma;h}$, $G_{rk}^{w;f}$, $G_{rij}^{w;h}$, $G_{pq}^{\tau;q}$, $G_{pqk}^{\tau;F}$, $G_r^{v,q}$ and $G_{rk}^{v,F}$. These eight Green's tensors are related such that we only need to determine one, for instance $G^{v;q}$ (see Table 2). It can be obtained by solving eq. (28) with the boundary condition (27).

(3) Four Green's tensors describe the case where source and observation points are in a solid, (D_s): $G_{pqij}^{\tau;h}$, $G_{pqk}^{\tau;f}$, $G_{rij}^{v;h}$ and $G_{rk}^{v;f}$. These four Green's tensors are related such that we only need to determine one, for instance $G^{v;f}$ (see Table 2). It is governed by the elastic wave equation

$$\rho_s^{(0)}(\mathbf{x})\,\partial_t^2 G_{kk'}^{v;f}(\mathbf{x}, t; \mathbf{x}') - \partial_m(C_{kmij}^{(0)}(\mathbf{x})\,\partial_i)G_{jk'}^{v;f}(\mathbf{x}, t; \mathbf{x}')$$

$$= \delta_{kk'}\,\delta(\mathbf{x}-\mathbf{x}_s)\partial_t[\delta(t)]\,, \quad \mathbf{x} \in D_s\,, \quad (29)$$

with the following boundary condition at the water/solid interface:

$$-\frac{1}{\rho_f}\,v_m(\mathbf{x})\partial_m G_k^{\sigma;f}(\mathbf{x}, t; \mathbf{x}') = v_m(\mathbf{x})\partial_t G_{mk}^{v;f}(\mathbf{x}, t; \mathbf{x}')\,, \quad \mathbf{x} \in A\,, \quad (30)$$

where $G_k^{\sigma;f}$ satisfies the following acoustic wave equation:

$$\frac{1}{v_P^2}\,\partial_t^2 G_k^{\sigma;f}(\mathbf{x}, t; \mathbf{x}') - \partial_m^2 G_k^{\sigma;f}(\mathbf{x}, t; \mathbf{x}') = 0, \quad \mathbf{x} \in D_f\,. \quad (31)$$

3.3 Scattering integral of equations of the particle velocity and pressure

Using the Green's tensors introduced above, we can estimate the scattered wavefield. Let us start with the receiver in the solid. For any receiver position $\mathbf{x}_r \in D_s$, the particle velocity and stress of the scattered wavefield can be expressed as

$$v_r^{(1)}(\mathbf{x}_r, t; \mathbf{x}_s) = \int_{\mathbf{x} \in \Omega_s} [G_{rk}^{v;f}(\mathbf{x}_r, t; \mathbf{x}) * f_k^{(1)}(\mathbf{x}, t; \mathbf{x}_s)$$

$$+ G_{rij}^{v;h}(\mathbf{x}_r, t; \mathbf{x}) * h_{ij}^{(1)}(\mathbf{x}, t; \mathbf{x}_s)]\,dV(\mathbf{x})\,, \quad (32)$$

$$\tau_{rs}^{(1)}(\mathbf{x}_r, t; \mathbf{x}_s) = \int_{\mathbf{x} \in \Omega_s} [G_{rsk}^{\tau;f}(\mathbf{x}_r, t; \mathbf{x}) * f_k^{(1)}(\mathbf{x}, t; \mathbf{x}_s)$$

$$+ G_{rsij}^{\tau;h}(\mathbf{x}_r, t; \mathbf{x}) * h_{ij}^{(1)}(\mathbf{x}, t; \mathbf{x}_s)]\,dV(\mathbf{x})\,, \quad (33)$$

where * denotes convolution with respect to time and $f_k^{(1)}$ and $h_{ij}^{(1)}$ are given by eqs (22) and (23), respectively.

For the point located just at the water/solid interface, the expressions of the particle velocity in (32) and that of the stress field in (24) can be substituted into the boundary conditions (33) to estimate the scattered acoustic pressure field and the particle velocity at the interface.

Let us now look at the case where receivers are in the water. For any receiver position $\mathbf{x}_r \in D_{\mathrm{f}}$, the particle velocity and acoustic pressure of the scattered wavefield can be expressed as

$$w_r^{(1)}(\mathbf{x}_r, t; \mathbf{x}_{\mathrm{s}}) = \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} [G_{rk}^{w;f}(\mathbf{x}_r, t; \mathbf{x}) * f_k^{(1)}(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}}) + G_{rij}^{w;h}(\mathbf{x}_r, t; \mathbf{x}) * h_{ij}^{(1)}(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})]\, dV(\mathbf{x}), \tag{34}$$

$$\sigma^{(1)}(\mathbf{x}_r, t; \mathbf{x}_{\mathrm{s}}) = \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} [G_k^{\sigma;f}(\mathbf{x}_r, t; \mathbf{x}) * f_k^{(1)}(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}}) + G_{ij}^{\sigma;h}(\mathbf{x}_r, t; \mathbf{x}) * h_{ij}^{(1)}(\mathbf{x}, t; \mathbf{x}_{\mathrm{s}})]\, dV(\mathbf{x}) . \tag{35}$$

For this OBS experiment, we concentrate on particle velocity in the solid and acoustic pressure in the water. To simplify the derivation a little, we will perform a Fourier transform with respect to time:

$$v_r^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} [G_{rk}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) f_k^{(1)}(\mathbf{x}, \omega; \mathbf{x}_{\mathrm{s}}) + G_{rij}^{v;h}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) h_{ij}^{(1)}(\mathbf{x}, \omega; \mathbf{x}_{\mathrm{s}})]\, dV(\mathbf{x}) , \tag{36}$$

$$\sigma^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} [G_k^{\sigma;f}(\mathbf{x}_r, \omega; \mathbf{x}) f_k^{(1)}(\mathbf{x}, \omega; \mathbf{x}_{\mathrm{s}}) + G_{ij}^{\sigma;h}(\mathbf{x}_r, \omega; \mathbf{x}) h_{ij}^{(1)}(\mathbf{x}, \omega; \mathbf{x}_{\mathrm{s}})]\, dV(\mathbf{x}) . \tag{37}$$

This system of integral equations is known as the Lippmann–Schwinger equations. It is usually solved iteratively, with the first iteration known as the Born approximation.

4 LINEARIZED FORWARD MODEL FOR OBS DATA

4.1 First Born approximation

The integral equations (45)–(46) for computing $\sigma^{(1)}$, $v_n^{(1)}$ are valid for any $\rho_{\mathrm{s}}^{(1)}$, $s_{ijpq}^{(1)}$, irrespective of their shape or size. We limit ourselves to a weak approximation case where we explicitly assume that $\rho_{\mathrm{s}}^{(1)}$ and $s_{ijpq}^{(1)}$ are small, which implies that the contrasts of volume source densities of force $f_k^{(1)}$ and deformation rate $h_{ij}^{(1)}$ can be approximated as follows:

$$f_k^{(1)} = -\rho_{\mathrm{s}}^{(1)}\, \partial_t [v_r^{(0)} + v_r^{(1)}] \approx -\rho^{(1)}\, \partial_t v_r^{(0)} , \tag{38}$$

$$h_{ij}^{(1)} = -s_{ijpq}^{(1)}\, \partial_t [\tau_{pq}^{(0)} + \tau_{pq}^{(1)} \approx -s_{ijpq}^{(1)}\, \partial_t \tau_{pq}^{(0)}] , \tag{39}$$

where the terms $o(\rho^{(1)}, v_r^{(1)})^2$ and $o(s_{ijpq}^{(1)}, \tau_{pq}^{(1)})^2$ are neglected.

4.2 Incident field

The computation of $f_k^{(1)}$ and $h_{ij}^{(1)}$ requires the incident fields $v_r^{(0)}$ and $\tau_{pq}^{(0)}$. These incident fields satisfy eqs (17), (18) and (19). They can be written as functions of the Green's tensors $G_r^{v;q}$ and $G_{pq}^{\tau;q}$. These incident fields can be written

$$v_r^{(0)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = G_r^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) a(\omega) , \tag{40}$$

$$\tau_{pq}^{(0)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = G_{pq}^{\tau;q}(\mathbf{x}_r, \omega; \mathbf{s}) a(\omega) . \tag{41}$$

4.3 Linearized forward problem: pressure and particle velocity

Inserting the incident fields (40) and (41) into (36) and (37), using the relations between the different Green's tensors described in Table 1 and invoking the property (14) between the compliances and stiffnesses of the inclusion, we arrive at the following solution:

$$v_r^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = -i\omega a(\omega) \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} \left[G_{rk}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x}) \rho_{\mathrm{s}}^{(1)}(\mathbf{x}) G_k^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) + \frac{1}{i\omega}\, \partial_p G_{qr}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x}) c_{pqij}^{(1)}(\mathbf{x}) \partial_i G_j^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) \right] dV , \tag{42}$$

$$\sigma^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_{\mathrm{s}}) = -i\omega a(\omega) \int_{\mathbf{x}\in\Omega_{\mathrm{s}}} \left[G_k^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) \rho_{\mathrm{s}}^{(1)}(\mathbf{x}) G_k^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) + \frac{1}{i\omega}\, \partial_p G_q^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) c_{pqij}^{(1)}(\mathbf{x}) \partial_i G_j^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) \right] dV . \tag{43}$$

Note that only two Green's tensors from the 16 described in Table 1 are needed for the computation of the scattered field of acoustic pressure and particle velocity: $G_k^{v;q}$ and $G_{rk}^{v;f}$. Fig. 2 illustrates the contributions of these Green's tensors in the construction of scattering events in OBS data. Note also that these scattered fields are now expressed as functions of $c_{pqij}^{(1)}$ instead of $s_{pqij}^{(1)}$ due to the property (14).

4.4 Linearized forward problem: *P*- and *S*-wave potentials

Sometimes it is useful to work with scattered *P*- and *S*-wave potentials instead of scattered particle velocities $v_j^{(1)}$. Here we describe equivalent equations for scattered *P*- and *S*-wave potentials.

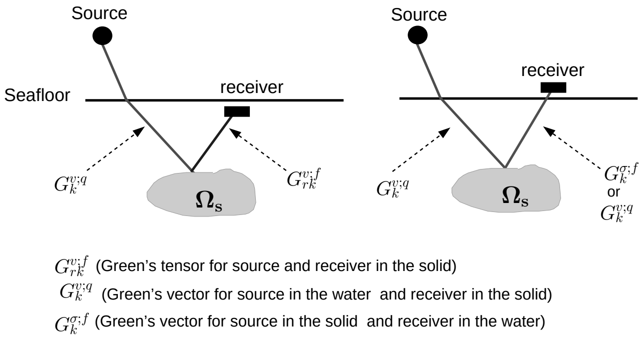

Figure 2. The final equations of the linearized forward problem for multicomponent OBS data can be expressed as functions of two Green's functions.

Let $v_j^{(1)}$ be the demultipled particle velocity. We can define the scattered P- and S-wave potential fields by applying the divergence and curl operators, respectively, to $v_j^{(1)}$, i.e.

$$\begin{aligned}\phi^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s) &= \frac{1}{-i\omega}\, \partial_j v_j^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s)\,,\\ \psi_j^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s) &= \frac{1}{-i\omega}\, \epsilon_{jkl}\partial_k v_l^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s)\,,\end{aligned} \tag{44}$$

where ϵ_{jkl} is the Levi–Civita tensor (the alternating tensor). Substituting the expression $v_j^{(1)}$ into eq. (43) leads to

$$\begin{aligned}\phi^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s) = a(\omega) \int_{\mathbf{x}\in\Omega_s} \Big[&\partial'_n G_{nk}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x})\rho^{(1)}(\mathbf{x}) G_k^{v;q}(\mathbf{x}_r, \omega; \mathbf{x})\\ &+ \frac{1}{-\omega^2}\, \partial'_n \partial_p G_{qn}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x}) c_{pqij}^{(1)}(\mathbf{x})\\ &\times \partial_i G_j^{v;q}(\mathbf{x}_r, \omega; \mathbf{x})\Big]\, dV,\end{aligned} \tag{45}$$

$$\begin{aligned}\psi_i^{(1)}(\mathbf{x}_r, \omega; \mathbf{x}_s) = a(\omega) \int_{\mathbf{x}\in\Omega_s} \Big[&\epsilon_{ijn}\partial'_n G_{jk}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x})\rho^{(1)}(\mathbf{x})\\ &\times G_k^{v;q}(\mathbf{x}_r, \omega; \mathbf{x}) + \frac{1}{-\omega^2}\, \epsilon_{ijn}\partial'_n \partial_p\\ &\times G_{qn}^{v;f}(\mathbf{x}_r, \omega; \mathbf{x}) c_{pqij}^{(1)}(\mathbf{x})\partial_i G_j^{v;q}(\mathbf{x}_r, \omega; \mathbf{x})\Big]\, dV,\end{aligned} \tag{46}$$

where the operator ∂'_n indicates partial differentiation with respect to receiver coordinates ($\mathbf{x}_r$), while ∂_n indicates partial differentiation with respect to coordinates of the generic point $\mathbf{x}$.

5 CONCLUSIONS

We have derived a rigorous solution to the linearized forward problem for multicomponent OBS data where the boundary conditions at a water/solid interface are explicitly taken into account. We have used the reciprocity theorem to reduce the dependence of our solutions on only two Green's tensors: the Green's tensor corresponding to a source in the water and a receiver in the solid, and that corresponding to both source and receiver in the solid.

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APPENDIX A: RELATIONSHIPS BETWEEN THE GREEN'S TENSORS INVOKED IN THE LINEARIZED FORWARD PROBLEM

Eqs (32), (33), (34), (35), (36) and (37), describing the scattering forward problem, invoke 16 Green's tensors. In fact, we only need three of them as the other 13 can be obtained from these by using the reciprocity theorem and the wave equations. Our objective in this Appendix is to establish these relationships.

This Appendix is organized into two sections. The first section deals with the relationships between the Green's tensors in Table 1, which are due to the reciprocity theorem, and those that are due to the wave equations.

A1 Reciprocity theorem

A1.1 Source and observation points in the water

In linear elastic media, the reciprocity theorem interrelates, in a specific manner, the quantities that characterize two different physical states that could occur in the same domain in space–time. If we denote these two states A and B, such that

in state A forces are $\{q^A, F_k^A\}$ and the wavefield is described by $\{\sigma^A, w_k^A\}$;

in state B forces are q^B, F_k^B and the wavefield is described by σ^B, w_k^B,

they can be related by the reciprocity theorem as follows:

$$\begin{aligned}&\int_{D_f} [F_k^{(A)}(\mathbf{x}, t, \mathbf{x}_A) w_k^{(B)}(\mathbf{x}, t, \mathbf{x}_B) + q^{(A)}(\mathbf{x}, t, \mathbf{x}_A)\sigma^{(B)}(\mathbf{x}, t, \mathbf{x}_B)]\, dV\\ &\quad= \int_{D_f} [F_r^{(B)}(\mathbf{x}, t, \mathbf{x}_B) w_r^{(A)}(\mathbf{x}, t, \mathbf{x}_A)\\ &\qquad + q^{(B)}(\mathbf{x}, t, \mathbf{x}_B)\sigma^{(A)}(\mathbf{x}, t, \mathbf{x}_A)]\, dV.\end{aligned} \tag{A1}$$

This formula is more simple than those encountered in Gangi (1970), de Hoop (1990) and Wapenaar & Fokkema (1999) because we have considered that the medium is unbounded (free-surface multiples are not taken into account).

We are interested in two particular states, A and B, which lead to one of the relations between the Green's tensors in Table 2. For state A, we choose

$$q^{(A)}(\mathbf{x}, t, \mathbf{x}_A) = a\, \delta(t)\delta(\mathbf{x}-\mathbf{x}_A) \text{ and } F_k^{(A)}(\mathbf{x}, t, \mathbf{x}_A) = 0\,, \tag{A2}$$

where a is an arbitrary constant scalar. This choice implies that

$$\sigma^{(A)} = G^{\sigma;q}(\mathbf{x}, t, \mathbf{x}_A)\, a \text{ and } w_r^{(A)} = G_r^{w;q}(\mathbf{x}, t, \mathbf{x}_A)\, a\,. \tag{A3}$$

For state B, we choose the opposite scenario, i.e.

$$q^{(B)}(\mathbf{x}, t, \mathbf{x}_B) = 0 \text{ and } F_k^{(B)}(\mathbf{x}, t, \mathbf{x}_B) = b_k\, \delta(t)\delta(\mathbf{x}-\mathbf{x}_B)\,, \tag{A4}$$

where b_k is an arbitrary constant vector. This choice implies that

$$\sigma^{(B)}=G_k^{\sigma;F}(\mathbf{x},\,t,\,\mathbf{x}_{\mathrm{B}})\,b_k \;\text{ and }\; w_r^{(B)}=G_{rk}^{w;F}(\mathbf{x},\,t,\,\mathbf{x}_{\mathrm{B}})\,b_k\,. \tag{A5}$$

By substituting (A2)–(A5) into (A1) and invoking the condition that the resulting equation has to hold for arbitrary a and b_k, we arrive at one of the relationships described in Table 2, namely

$$G_k^{\sigma;F}(\mathbf{x}_{\mathrm{A}},\,t,\,\mathbf{x}_{\mathrm{B}})=G_k^{w;q}(\mathbf{x}_{\mathrm{B}},\,t,\,\mathbf{x}_{\mathrm{A}})\,. \tag{A6}$$

A1.2 Source and observation points in the solid

A derivation similar to that of the previous case where the two states A and B are such that

in state A forces are h_{ij}^A, f_k^A and the wavefield is described by τ_{pq}^A, v_k^A;

in state B forces are h_{ij}^B, f_k^B and the wavefield is described by τ_{pq}^A, v_k^B

leads to

$$G_{rij}^{v;h}(\mathbf{x}_{\mathrm{A}},\,t,\,\mathbf{x}_{\mathrm{B}})=G_{ijr}^{\tau;f}(\mathbf{x}_{\mathrm{B}},\,t,\,\mathbf{x}_{\mathrm{A}})\,. \tag{A7}$$

A1.3 Source point in the water and observation point in the solid or vice versa

A derivation similar to that of the previous case where the two states A and B are such that

in state A forces are q^A, F_k^A and the wavefield is described by τ_{pq}^A, v_k^A;

in state B forces are h_{ij}^B, f_k^B and the wavefield is described by σ_{pq}^A, w_k^B,

leads to

$$G_k^{\sigma;f}(\mathbf{x}_{\mathrm{A}},\,t,\,\mathbf{x}_{\mathrm{B}})=G_k^{v;q}(\mathbf{x}_{\mathrm{B}},\,t,\,\mathbf{x}_{\mathrm{B}}) \tag{A8}$$

and

$$G_{rij}^{w;h}(\mathbf{x}_{\mathrm{A}},\,t,\,\mathbf{x}_{\mathrm{B}})=G_{ijr}^{\tau;F}(\mathbf{x}_{\mathrm{B}},\,t,\,\mathbf{x}_{\mathrm{A}})\,. \tag{A9}$$

A2 Relationships between Green's tensors based on the wave equations

Some of the relationships in Table 2 are not based on the reciprocity theorem but rather on the wave equations. To derive them, we need to introduce Green's tensors associated with the reference medium $\mathbf{m}^{(0)}$. They are solutions to the coupled equations (1)–(4) for four particular cases of a source system: (i) the case where the water volume source density is a unit impulse, localized in space and time, with $F_k=f_{k'}=h_{ij}=0$; (ii) the case where the water volume source density of force F_k is a unidirectional unit impulse, localized in space and time, with $q=f_{k'}=h_{ij}=0$; (iii) the case where the solid volume of source density of force f_k is a unidirectional unit impulse, localized in space and time, with $q=F_k=h_{ij}=0$; and (iv) the case where the deformation rate h_{ij} is a monopole unit impulse, localized in space and time, with $q=F_k=f_{k'}=0$.

A2.1 Case I.

The water volume source density in D_{f} is a unit impulse, localized in space and time, with $F_k=f_{k'}=h_{ij}=0$. The Green's tensors in Table 1 with a source point in water satisfy the following equations:

$$\left.\begin{array}{l} \partial_k G^{\sigma;q}(\mathbf{x},\,t,\,\mathbf{x}')+\rho_{\mathrm{f}}\,\partial_t G_r^{w;q}(\mathbf{x},\,t,\,\mathbf{x}')=\delta(t)\delta(\mathbf{x}-\mathbf{x}') \\ \partial_r G_r^{w;q}(\mathbf{x},\,t,\,\mathbf{x}')+\kappa\,\partial_t G^{\sigma;q}(\mathbf{x},\,t,\,\mathbf{x}_{\mathrm{s}})=0 \end{array}\right\} ,\quad \mathbf{x}\in D_{\mathrm{f}}\,, \tag{A10}$$

$$\left.\begin{array}{l} -\Delta_{kmpq}\partial_m G_{pq}^{\tau;q}(\mathbf{x},\,t,\,\mathbf{x}')+\rho^{(0)}(\mathbf{x})\partial_t G_r^{v;q}(\mathbf{x},\,t,\,\mathbf{x}')=0 \\ \Delta_{ijmr}\partial_m G_r^{v;q}(\mathbf{x},\,t,\,\mathbf{x}')-s_{ijpq}^{(0)}(\mathbf{x})\partial_t G_{pq}^{\tau;q}(\mathbf{x},\,t,\,\mathbf{x}')=0 \end{array}\right\} ,\quad \mathbf{x}\in D_{\mathrm{s}}\,. \tag{A11}$$

We can deduce from these equations two of the relationships in Table 2, namely

$$\partial_r G_r^{w;q}(\mathbf{x},\,t,\,\mathbf{x}')=-\kappa\,\partial_t G^{\sigma;q}(\mathbf{x},\,t,\,\mathbf{x}')\,, \tag{A12}$$

and

$$\partial_t G_{pq}^{\tau;q}(\mathbf{x},\,t,\,\mathbf{x}')=c_{pqij}^{(0)}(\mathbf{x})\partial_i G_j^{v;q}(\mathbf{x},\,t,\,\mathbf{x}')\,. \tag{A13}$$

A2.2 Case II

In this case the water volume source density of force F_k is a unidirectional unit impulse, localized in space and time, with $q=f_{k'}=h_{ij}=0$. The Green's tensors in Table 1 with a source point in water satisfy the following equations:

$$\left.\begin{array}{l} \partial_k G_{k'}^{\sigma;F}(\mathbf{x},\,t,\,\mathbf{x}')+\rho_{\mathrm{f}}\,\partial_t G_{rk'}^{w;F}(\mathbf{x},\,t,\,\mathbf{x}')=0 \\ \partial_r G_{rk'}^{w;F}(\mathbf{x},\,t,\,\mathbf{x}')+\kappa\,\partial_t G_{k'}^{\sigma;f}(\mathbf{x},\,t,\,\mathbf{x}_{\mathrm{s}})=\delta_{kk'}\delta(t)\delta(\mathbf{x}-\mathbf{x}') \end{array}\right\} ,\quad \mathbf{x}\in D_{\mathrm{f}}\,, \tag{A14}$$

$$\left.\begin{array}{l} -\Delta_{kmpq}\partial_m G_{pqk}^{\tau;F}(\mathbf{x},\,t,\,\mathbf{x}')+\rho^{(0)}(\mathbf{x})\partial_t G_{rk}^{v;F}(\mathbf{x},\,t,\,\mathbf{x}')=0 \\ \Delta_{ijmr}\partial_m G_{rk}^{v;F}(\mathbf{x},\,t,\,\mathbf{x}')-s_{ijpq}^{(0)}(\mathbf{x})\partial_t G_{pqk}^{\tau;F}(\mathbf{x},\,t,\,\mathbf{x}')=0 \end{array}\right\} ,\quad \mathbf{x}\in D_{\mathrm{s}}\,. \tag{A15}$$

We can deduce from these equations two of the relationships in Table 2, namely

$$\partial_r G_{rk}^{w;F}(\mathbf{x},\,t,\,\mathbf{x}')=-\kappa\,\partial_t G_k^{\sigma;F}(\mathbf{x},\,t,\,\mathbf{x}')\,,\quad \mathbf{x}\neq\mathbf{x}'\,, \tag{A16}$$

and

$$\partial_t G_{pqk}^{\tau;F}(\mathbf{x},\,t,\,\mathbf{x}')=c_{pqij}^{(0)}(\mathbf{x})\partial_i G_{ik}^{v;F}(\mathbf{x},\,t,\,\mathbf{x}')\,. \tag{A17}$$

A2.3 Case III

This is a case where the solid volume of source density of force f_k is a unidirectional unit impulse, localized in space and time, with $q=F_k=h_{ij}=0$. The Green's tensors in Table 1 with a source point in the solid satisfy the following equations:

$$\begin{array}{l}\partial_k G^{\sigma;f}_{k'}(\mathbf{x},\, t,\, \mathbf{x}') + \rho_{\mathrm{f}}\, \partial_t G^{w;f}_{rk'}(\mathbf{x},\, t,\, \mathbf{x}') = 0 \\ \partial_r G^{w;f}_{rk'}(\mathbf{x},\, t,\, \mathbf{x}') + \kappa\, \partial_t G^{\sigma;f}_{k'}(\mathbf{x},\, t,\, \mathbf{x}_{\mathrm{s}}) = 0\end{array} ,\quad \mathbf{x} \in D_{\mathrm{f}}\,, \tag{A18}$$

$$\begin{array}{l}-\Delta_{kmpq}\partial_m G^{\tau;f}_{pqk}(\mathbf{x}, t, \mathbf{x}') + \rho^{(0)}(\mathbf{x})\partial_t G^{v;f}_{rk}(\mathbf{x}, t, \mathbf{x}') = \delta_{kk'}\delta(t)\delta(\mathbf{x}-\mathbf{x}') \\ \Delta_{ijmr}\partial_m G^{v;f}_{rk}(\mathbf{x},\, t,\, \mathbf{x}') - s^{(0)}_{ijpq}(\mathbf{x})\partial_t G^{\tau;f}_{pqk}(\mathbf{x},\, t,\, \mathbf{x}') = 0\end{array} ,$$

$$\mathbf{x} \in D_{\mathrm{s}}\,. \tag{A19}$$

We can deduce from these equations two of the relationships in Table 2, namely

$$\partial_r G^{w;f}_{rk}(\mathbf{x},\, t,\, \mathbf{x}') = -\kappa\, \partial_t G^{\sigma;f}_{k}(\mathbf{x},\, t,\, \mathbf{x}') \tag{A20}$$

and

$$\partial_t G^{\tau;f}_{pqk}(\mathbf{x},\, t,\, \mathbf{x}') = c^{(0)}_{pqij}(\mathbf{x})\partial_i G^{v;f}_{jk}(\mathbf{x},\, t,\, \mathbf{x}')\,. \tag{A21}$$

A2.4 Case IV

This is a case where the deformation rate h_{ij} is a monopole unit impulse, localized in space and time, with $q = F_k = f_{k'} = 0$. The Green's tensors in Table 1 with the source point in a solid satisfy the following equations:

$$\begin{array}{l}\partial_k G^{\sigma;h}_{ij}(\mathbf{x},\, t,\, \mathbf{x}') + \rho_{\mathrm{f}}\, \partial_t G^{w;h}_{rij}(\mathbf{x},\, t,\, \mathbf{x}') = 0 \\ \partial_r G^{w;h}_{rij}(\mathbf{x},\, t,\, \mathbf{x}') + \kappa\, \partial_t G^{\sigma;h}_{ij}(\mathbf{x},\, t,\, \mathbf{x}_{\mathrm{s}}) = 0\end{array} ,\quad \mathbf{x} \in D_{\mathrm{f}}\,, \tag{A22}$$

$$\begin{cases}-\Delta_{kmpq}\partial_m G^{\tau;h}_{pqij}(\mathbf{x},\, t,\, \mathbf{x}') + \rho^{(0)}(\mathbf{x})\partial_t G^{v;h}_{rk}(\mathbf{x},\, t,\, \mathbf{x}') = 0 \\ \Delta_{ijmr}\partial_m G^{v;h}_{rij}(\mathbf{x}, t, \mathbf{x}') - s^{(0)}_{ijpq}(\mathbf{x})\partial_t G^{\tau;f}_{pqij}(\mathbf{x}, t, \mathbf{x}') = \Delta_{iji'j'}\delta(t)\delta(\mathbf{x}-\mathbf{x}')\end{cases} ,$$

$$\mathbf{x} \in D_{\mathrm{s}}\,. \tag{A23}$$

We can deduce from these equations two of the relationships in Table 2, namely

$$\partial_r G^{w;h}_{rij}(\mathbf{x},\, t,\, \mathbf{x}') = -\kappa\, \partial_t G^{\sigma;h}_{ij}(\mathbf{x},\, t,\, \mathbf{x}') \tag{A24}$$

and

$$\partial_t G^{\tau;h}_{pqij}(\mathbf{x},\, t,\, \mathbf{x}') = c^{(0)}_{pqmr}(\mathbf{x})\partial_m G^{v;f}_{rij}(\mathbf{x},\, t,\, \mathbf{x}')\,,\quad \mathbf{x} \neq \mathbf{x}'\,. \tag{A25}$$